Interaction of fast magnetoacoustic solitons in dense plasmas

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Abstract

One dimensional propagation of fast magnetoacoustic solitary waves in dense plasmas with degenerate electrons is investigated in this paper in the small amplitude limit. In this regard, Korteweg de-Vries (KdV) equation is derived and discussed using the plasma parameters that are typically found in white dwarf stars. The interaction of fast magnetoacoustic solitons is explored by using the Hirota bilinear formalism which admits multi soliton solutions. It is observed that the values of the propagation vectors determine the interaction of solitary waves. It is further noted that the amplitude of the respective solitary waves remain unchanged after the interaction, however, they do experience a phase shift.

Magnetoacoustic Waves

Magnetoacoustic waves are amongst the three normal modes of a magnetized plasma from the magnetohydrodynamics stand point besides the acoustic and Alfven waves.
Dispersion relation for magnetoacoustic waves is:

$$\frac{\omega^2}{k^2} = \frac{1}{2}(C_s^2 + v_A^2) \pm \frac{1}{2}\sqrt{(C_s^2+v_A^2) - 4C_s^2 v_A^2 \cos^2\theta}$$

Applications of Magnetoacoustic Waves

The **formation of jets** such as dynamic fibrils, mottles, and spicules in the solar chromosphere is a natural consequence of upwardly propagating slow-mode magnetoacoustic shocks (ApJ 647: L73-L76, 2006).

It has been shown that large-amplitude MHD shocks in low-β regions could be a viable mechanism for **coronal heating and wind acceleration** in regions of open magnetic field lines (ApJ, 596:646-655, 2003).

It has been shown that **shock heating** by slow magnetosonic waves is expected to be relevant at most heights in solar coronal plumes, although slow magnetosonic waves are most likely not a solely operating energy supply mechanism (ApJ, 549: L143-L146, 2001).

The magnetosonic waves are believed to be responsible for **particle acceleration** in the Earth's magnetosphere and **plasma heating** in the solar atmosphere. They are also used in the fusion devices for plasma heating.

Mathematical Model

Magnetized Electron-ion quantum plasma.

$B=(0,0,B_0)$ is Magnetic field.

$E=(E_x,E_y,0)$ is the Electric field.

Propagation lies along x-axis.

Normalized continuity and momentum equations for ion are

$$\frac{\partial n_e}{\partial t} + \nabla\cdot(n_e\mathbf{v}_e) = 0, \qquad [\frac{\partial}{\partial t} + \mathbf{v}_i\cdot\nabla]\mathbf{v}_i = e[\mathbf{E} + \mathbf{v}_i\times\mathbf{B}].$$

The continuity and momentum equations for electron dynamics are

$$\frac{\partial n_i}{\partial t} + \nabla\cdot(n_i\mathbf{v}_i) = 0, \qquad m_e n_e[\frac{\partial}{\partial t} + \mathbf{v}_e\cdot\nabla]\mathbf{v}_e = -en_e(\mathbf{E} + \mathbf{v}_e\times\mathbf{B}) - \nabla P_e.$$

The quantum pressure is considered by using quantum statistics

$$P_e = \frac{\hbar^2(3\pi^2)^{2/3}}{5m_e} n_e^{5/3}$$

The normalized momentum equation for electrons, then becomes

$$\rho[\frac{\partial}{\partial t} + \mathbf{v}_e\cdot\nabla]\mathbf{v}_e = -(\mathbf{E} + \mathbf{v}_e\times\mathbf{B}) - \beta\nabla n_e^{\frac{2}{3}}$$

$\beta = C_s^2/V_A^2$ is the ratio of Fermi and magnetic pressures,

The normalized Maxwell's equations

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial T}, \qquad \nabla\times\mathbf{B} = n_i\mathbf{v}_i - n_e\mathbf{v}_e.$$

$$B\to\frac{B}{B_0}, n\to\frac{n}{n_0}, V\to\frac{V}{V_A}, E\to\frac{E}{V_A B_0}, t\to\frac{t}{\omega_{ci}^{-1}}, x\to\frac{x}{V_A/\omega_{ci}}.$$

Korteweg de Vries (KdV) Equation

To study the magnetoacoustic perturbations propagating in dense plasma, stretch the independent variables as

$$\xi = \epsilon^{1/2}(x - v_0 t), \qquad \tau = \epsilon^{3/2} t,$$

where ϵ is a small parameter measuring weakness of nonlinearity and v_0 is wave phase velocity normalized to v_A.

The perturbed quantities are expanded as

$$n_l = 1 + \epsilon n_l^{(1)} + \epsilon^2 n_l^{(2)} + \epsilon^3 n_l^{(3)} + \cdots,$$
$$v_{lx} = 0 + \epsilon v_{lx}^{(1)} + \epsilon^2 v_{lx}^{(2)} + \epsilon^3 v_{lx}^{(3)} + \cdots,$$
$$v_{ly} = 0 + \epsilon^{3/2} v_{ly}^{(1)} + \epsilon^{5/2} v_{ly}^{(2)} + \epsilon^{7/2} v_{ly}^{(3)} + \cdots,$$
$$B_z = 1 + \epsilon B_z^{(1)} + \epsilon^2 B_z^{(2)} + \epsilon^3 B_z^{(3)} + \cdots,$$
$$E_x = \epsilon^{3/2} E_x^{(1)} + \epsilon^{5/2} E_x^{(2)} + \epsilon^{7/2} E_x^{(3)} + \cdots,$$
$$E_y = \epsilon E_y^{(1)} + \epsilon^2 E_y^{(2)} + \epsilon^3 E_y^{(3)} + \cdots,$$

The dispersion relation is then

$$v_0 = \sqrt{\frac{1+\frac{2}{3}\beta}{1+\rho}}$$

Algebraic manipulation of higher order eqns of ϵ leads to KdV equation.

$$\frac{\partial B_z^{(1)}}{\partial\tau} + AB_z^{(1)}\frac{\partial B_z^{(1)}}{\partial\xi} + B\frac{\partial^3 B_z^{(1)}}{\partial\xi^3} = 0,$$

where coefficient of nonlinearity ond dispersion are

$$A = \frac{3 + (16/9)\beta}{2v_0(1+\rho)},$$
$$B = \frac{1}{2v_0(1+\rho)}\left[v_0^4 + \rho(v_0^2\rho - \frac{2}{3}\beta)^2 - \frac{1}{(1+\rho)}\{v_0^2(1-\rho^2) + \frac{2}{3}\beta\rho\}^2\right]$$

Multi Soliton Solution of KdV equation

Solution of KdV equation is found by following the **Hirota Bilinear formalism** to transform a nonlinear differential equations into linear differential equations through a change of dependent variable

We assume that KdV equation possesses a solution of the form $v^{(1)}(\xi,\tau) = \frac{12B}{A}\frac{\partial^2[\log f(\xi,\tau)]}{\partial\xi^2}$

We use this transformation to form bilinear equation $f_{\xi,\tau}f - f_\xi f_\tau + f_{\xi,\xi,\xi,\xi}f - 4f_{\xi,\xi,\xi}f_\xi + 3f_{\xi,\xi}^2 = 0$

This bilinear equation can be written in terms of the Hirota-D operators as

$$D_\xi(D_\tau + BD_\xi^3)(f.f) = 0.$$

Here

$$D_\xi^l D_\tau^m(\sigma.\sigma) = (\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\xi'})^l(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\tau'})^m(\sigma(\xi,\tau)\sigma(\xi',\tau'))\Big|_{\xi=\xi',\tau=\tau'}$$

To find the multi-soliton solution, perturbation method is used to expand $f(\xi,\tau)$ in terms of power series of ε as $f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \ldots$

This expansion is substituted in the Hirota bilinear equation and terms with different orders of ε are collected.

The solution describing a single soliton is $f_1 = e^\eta$, where $\eta = k\xi + \omega\tau + \gamma$,

ω is found by substituting expansion of $f(\xi,\tau)$ in the ε-order of bilinear form of equation, which gives $\omega = -Bk^3$.

Substitution of $f_1 = Exp[\eta]$ gives $f_j = 0$ for $j = 2,3,4,\ldots$

Thus the **one-soliton solution of KdV** equation is $B_z^{(1)}(\xi,\tau) = \frac{12B}{A}\frac{\partial^2[\log(1+\exp(k\xi - Bk^3\tau + \gamma))]}{\partial\xi^2}$

$$\Rightarrow B_z^{(1)}(\xi,\tau) = \frac{3B}{A}k^2\,\mathrm{sech}^2(\frac{k\xi - Bk^3\tau + \gamma}{2}).$$

Two soliton solution of this non linear equation are found by choosing

$$f_l = e^{\eta_1} + e^{\eta_2} \text{ where } \eta_i = k_i\xi - \omega_i\tau + \gamma_i, \ (i = 1,2).$$

Substituting this solution in ε–order equation gives the dispersion relation $\omega_i = -Bk_i^3$.
For ε^2–order equation, we take $f_2 = a_{12}e^{\eta_1+\eta_2}$

which give the interaction parameter $a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}$

Putting f_1 and f_2 gives all remaining $f_j = 0$. This gives the two solitons solution as:

$$f(\xi,\tau) = 1 + e^{\eta_1} + e^{\eta_2} + a_{12}e^{\eta_1+\eta_2}$$

$$B_z^{(1)}(\xi,\tau) = \frac{12B}{A}\frac{\partial^2[\log(1+e^{\eta_1}+e^{\eta_2}+a_{12}e^{\eta_1+\eta_2})]}{\partial\xi^2}$$

Numerical Results of 1-Soliton

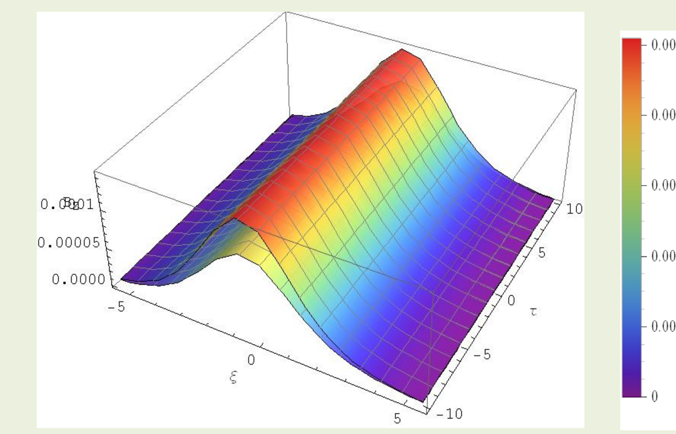


Fig 1: Effect of density on the solitary wave solution of KdV equation. Upper curve is for $n_0=10^{33}m^{-3}$ whereas lower curve is for $n_0=5\times10^{33}m^{-3}$. Other parameters are $B_0=10^6$T, $T=10^6$K.

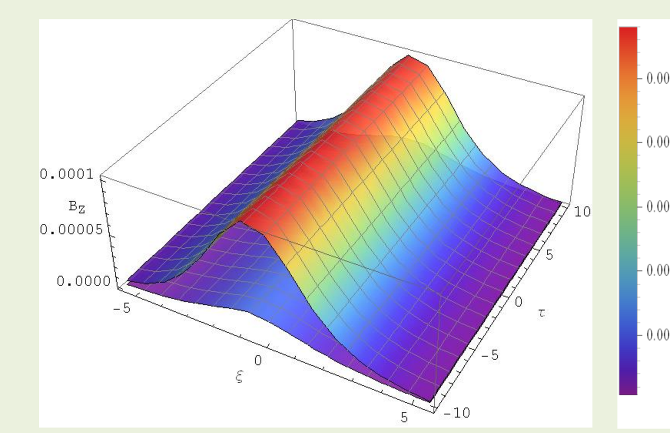
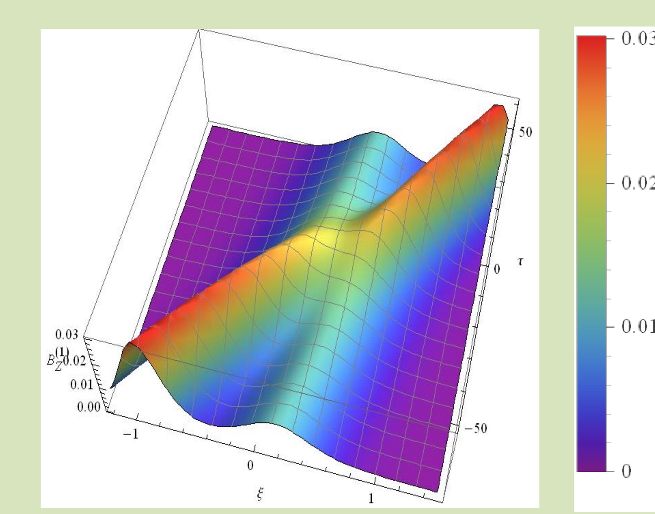
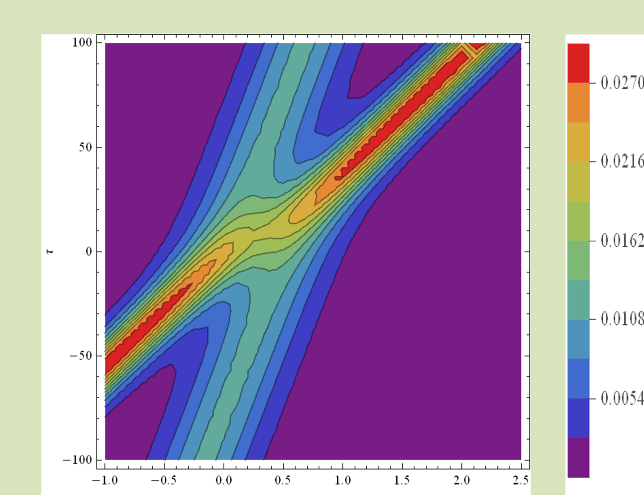


Fig 2: Effect of magnetic field on the solitary wave solution of KdV equation. Upper curve is for $B_0=10^6$T whereas lower curve is for $B_0=10^7$T. Other parameters are $n_0=10^{33}m^{-3}$, $T=10^6$K.

Interaction of two Solitons

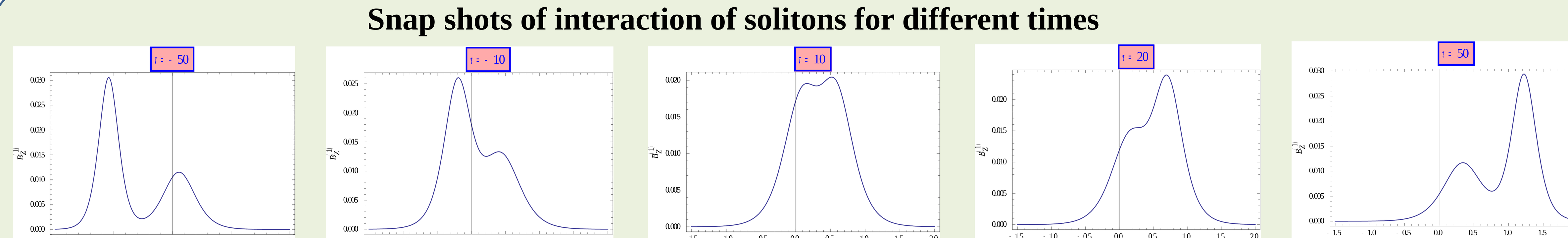


Variation of two soliton solution $B_z^{(1)}(\xi,\tau)$ versus ξ and τ with $n_0=10^{33}m^{-3}$, $B_0=10^6$T, $k_1=9$, $k_2=5.5$, and $T=10^6$K.



Parametric Plot for the variation of two soliton solution $B_z^{(1)}(\xi,\tau)$ versus ξ and τ.

Snap shots of interaction of solitons for different times



Results and Discussion

- We have investigated the Magnetoacoustic solitary wave structures in degenerate dense astrophysical electron-ion plasmas. For the mentioned problem we have studied the one-soliton and two-soliton solutions.
- It is shown that a slight increase in the density decreases the amplitude of the soliton.
- It is also observed that a change in the magnetic field for a fixed value of density increases the amplitude of the solitons.
- We have evaluated the exact solution of two solitons of KdV by Hirota bilinear formalism.
- The two solitons interact elastically with each other, so that the individual solitons retain their shape.
- The superposition principle does not hold for the interaction of these non linear waves.
- The solitons of same amplitude will never interact.

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